Commun. Combin., Cryptogr. & Computer Sci., 2 (2023), 149–155

ISSN:2783-5456

Communications in Combinatorics, Cryptography & Computer Science

Journal Homepage: http://cccs.sgh.ac.ir

Integral inequalities for differentiable (h, m)-convex functions with generalized Caputo-type derivatives

Paulo Matias Guzmán[a,]

[a]guzmanpaulomatias@gmail.com

Abstract

In this work we obtain integral inequalities of the Hermite-Hadamard type, using generalized derivatives of the Caputo type. Throughout the work, we see that several results reported in the literature are particular cases of those presented here.

Keywords: Integral inequalities, (h, m)-convex functions, Generalized Caputo type derivatives.

2020 MSC: 26A33, 26D10, 47A63

1. Introduction

In Mathematics, the notion of convex function plays a very prominent role, due to its multiple applications and its theoretical overlaps with various other areas of science.

One of the most important inequalities for convex functions is the well-known Hermite-Hadamard inequality:

$$\chi\left(\frac{\rho_1+\rho_2}{2}\right) \leqslant \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\chi(x)dx \leqslant \frac{\chi(\rho_1)+\chi(\rho_2)}{2}.$$

This inequality holds for any convex χ function on the interval $[\rho_1, \rho_2]$. Gives an estimate of the mean value of a convex function.

The following two definitions have been presented in [1]:

Definition 1.1. Let $h : [0, 1] \to \mathbb{R}$ be a non-negative function, $h \neq 0$ and $\chi : I = [0, +\infty) \to [0, +\infty)$. If the inequality

$$\chi(\mu\gamma + m(1-\mu)\sigma) \leqslant h^s(\mu)\chi(\gamma) + m(1-h^s(\mu))\chi\left(\frac{\sigma}{m}\right)$$

holds for all $\gamma, \sigma \in I$ and $\mu \in [0, 1]$, where $m \in [0, 1]$, $s \in [-1, 1]$, then the function χ is called (h, m)-modified convex of the first type in I.

*Corresponding author

Email address: guzmanpaulomatias@gmail.com (Paulo Matias Guzmán)

Received: November 3, 2023 Revised: November 10, 2023 Accepted: November 19, 2023

Definition 1.2. Let $h : [0,1] \to \mathbb{R}$ be a non-negative function, $h \neq 0$ and $\chi : I = [0,+\infty) \to [0,+\infty)$. If the inequality holds

$$\chi(\mu\gamma + m(1-\mu)\sigma) \leqslant h^s(\mu)\chi(\gamma) + m(1-h(\mu))^s \chi\left(\frac{\sigma}{m}\right)$$

for all $\gamma, \sigma \in I$ and $\mu \in [0,1]$, where $m \in [0,1]$, $s \in [-1,1]$, then the function χ is called (h,m)-modified convex of the second type in I.

The differential operators that we will use in our work are the following:

Definition 1.3. Let $\alpha > 0$, and $\alpha \neq 1,2,3,\dots$, $n = [\alpha]+1$, $f \in AC^n[a,b]$, the space of functions that have the n-th absolutely continuous derivatives. The weight Caputo derivatives of the right-hand side and the left-hand side of order α are defined as follows:

$$\left({}^C_n D^{w'}_{\rho_1+} f\right)(\rho_2) = \int_{\rho_1}^{\rho_2} w'\left[\frac{\rho_2 - x}{\frac{\rho_2-\rho_1}{r+1}}\right] f^{(n)}(x)dx,$$

$$\left({}^C_n D^{w'}_{\rho_2-} f\right)(\rho_1) = \int_{\rho_1}^{\rho_2} w'\left[\frac{x - \rho_1}{\frac{\rho_2-\rho_1}{r+1}}\right] f^{(n)}(x)dx.$$

In this work we obtain different variants of the Hermite-Hadamard inequality, within the framework of modified (h,m)-convex functions, using generalized operators.

2. Main Results

We present a version of the Hermite-Hadamard inequality using generalized Caputo-type derivatives.

Theorem 2.1. *Let χ be a positive function such that $\chi \in C^n[\rho_1,\rho_2]$. If $\chi^{(n)}$ is a modified (h,m)-convex function of the second type with $m \in (0,1]$ and $0 < \rho_1 < m\rho_2 < +\infty$, then:*

$$\begin{aligned}
&\chi^{(n)}\left(\frac{\rho_1+\rho_2}{2}\right)\int_0^1 w(\theta)d\theta \qquad (2.1)\\
\leqslant\ & h^s\left(\frac{1}{2}\right)\frac{(r+1)}{(\rho_2-\rho_1)}\left({}^C_n D^{w}_{\left(\frac{\rho_1+r\rho_2}{r+1}\right)+}\chi\right)(\rho_2) + \left(1-h\left(\frac{1}{2}\right)\right)^s \frac{(r+1)}{(\rho_2-\rho_1)}\left({}^C_n D^{w}_{\left(\frac{r\rho_1+\rho_2}{r+1}\right)-}\chi\right)(\rho_1)\\
\leqslant\ & \left[h^s\left(\frac{1}{2}\right)\chi^{(n)}(\rho_1) + \left(1-h\left(\frac{1}{2}\right)\right)^s \chi^{(n)}(\rho_2)\right]\int_0^1 w(\theta)h^s\left(\frac{\theta}{r+1}\right)d\theta\\
+\ & m\left[h^s\left(\frac{1}{2}\right)\chi^{(n)}\left(\frac{\rho_2}{m}\right) + \left(1-h\left(\frac{1}{2}\right)\right)^s \chi^{(n)}\left(\frac{\rho_1}{m}\right)\right]\int_0^1 w(\theta)\left(1-h\left(\frac{r+1-\theta}{r+1}\right)\right)^s d\theta.
\end{aligned}$$

Proof. Being $x, y \in [0,+\infty)$, $\theta = \frac{1}{2}$ and $m = 1$, we have

$$\chi^{(n)}\left(\frac{x+y}{2}\right) \leqslant h^s\left(\frac{1}{2}\right)\chi^{(n)}(x) + \left(1-h\left(\frac{1}{2}\right)\right)^s \chi^{(n)}(y).$$

Making $x = \frac{\theta}{r+1}\rho_1 + \left(\frac{r+1-\theta}{r+1}\right)\rho_2$, $y = \frac{\theta}{r+1}\rho_2 + \left(\frac{r+1-\theta}{r+1}\right)\rho_1$, with $\theta \in [0,1]$, we have

$$\begin{aligned}
\chi^{(n)}\left(\frac{\rho_1+\rho_2}{2}\right) \leqslant\ & h^s\left(\frac{1}{2}\right)\chi^{(n)}\left(\frac{\theta}{r+1}\rho_1 + \left(\frac{r+1-\theta}{r+1}\right)\rho_2\right) \qquad (2.2)\\
+\ & \left(1-h\left(\frac{1}{2}\right)\right)^s \chi^{(n)}\left(\frac{\theta}{r+1}\rho_2 + \left(\frac{r+1-\theta}{r+1}\right)\rho_1\right).
\end{aligned}$$

Multiplying the previous inequality by $w(\theta)$, to then integrate with respect to the variable θ between 0 and 1, and changing the variables, gives us the first inequality of (2.1).

$$\begin{aligned}
\chi^{(n)}\left(\frac{\rho_1+\rho_2}{2}\right)\int_0^1 w(\theta)d\theta &\leqslant h^s\left(\frac{1}{2}\right)\int_0^1 w(\theta)\chi^{(n)}\left(\frac{\theta}{r+1}\rho_1+\left(\frac{r+1-\theta}{r+1}\right)\rho_2\right)d\theta \\
&+\left(1-h\left(\frac{1}{2}\right)\right)^s\int_0^1 w(\theta)\chi^{(n)}\left(\frac{\theta}{r+1}\rho_2+\left(\frac{r+1-\theta}{r+1}\right)\rho_1\right)d\theta, \\
\chi^{(n)}\left(\frac{\rho_1+\rho_2}{2}\right)\int_0^1 w(\theta)d\theta &\leqslant h^s\left(\frac{1}{2}\right)\frac{(r+1)}{(\rho_1-\rho_2)}\int_{\rho_2}^{\frac{\rho_1+r\rho_2}{r+1}} w\left[\frac{(x-\rho_2)}{\frac{\rho_1-\rho_2}{(r+1)}}\right]\chi^{(n)}(x)\,dx \\
&+\left(1-h\left(\frac{1}{2}\right)\right)^s\frac{(r+1)}{(\rho_2-\rho_1)}\int_{\rho_1}^{\frac{r\rho_1+\rho_2}{r+1}} w\left[\frac{(x-\rho_1)}{\frac{(\rho_2-\rho_1)}{(r+1)}}\right]\chi^{(n)}(x)\,dx, \\
\chi^{(n)}\left(\frac{\rho_1+\rho_2}{2}\right)\int_0^1 w(\theta)d\theta &\leqslant h^s\left(\frac{1}{2}\right)\frac{(r+1)}{(\rho_2-\rho_1)}\int_{\frac{\rho_1+r\rho_2}{r+1}}^{\rho_2} w\left[\frac{(\rho_2-x)}{\frac{(\rho_2-\rho_1)}{(r+1)}}\right]\chi^{(n)}(x)\,dx \\
&+\left(1-h\left(\frac{1}{2}\right)\right)^s\frac{(r+1)}{(\rho_2-\rho_1)}\int_{\rho_1}^{\frac{r\rho_1+\rho_2}{r+1}} w\left[\frac{(x-\rho_1)}{\frac{(\rho_2-\rho_1)}{(r+1)}}\right]\chi^{(n)}(x)\,dx, \\
\chi^{(n)}\left(\frac{\rho_1+\rho_2}{2}\right)\int_0^1 w(\theta)d\theta &\leqslant h^s\left(\frac{1}{2}\right)\frac{(r+1)}{(\rho_2-\rho_1)}\left({}^{C}_{n}D^{w}_{\left(\frac{\rho_1+r\rho_2}{r+1}\right)+}\chi\right)(\rho_2) \\
&+\left(1-h\left(\frac{1}{2}\right)\right)^s\frac{(r+1)}{(\rho_2-\rho_1)}\left({}^{C}_{n}D^{w}_{\left(\frac{r\rho_1+\rho_2}{r+1}\right)-}\chi\right)(\rho_1).
\end{aligned}$$

We obtain from the right side of (2.2) the following

$$\begin{aligned}
&h^s\left(\frac{1}{2}\right)\chi^{(n)}\left(\frac{\theta}{r+1}\rho_1+\left(\frac{r+1-\theta}{r+1}\right)\rho_2\right)+\left(1-h\left(\frac{1}{2}\right)\right)^s\chi^{(n)}\left(\frac{\theta}{r+1}\rho_2+\left(\frac{r+1-\theta}{r+1}\right)\rho_1\right)= \\
&h^s\left(\frac{1}{2}\right)\chi^{(n)}\left(\frac{\theta}{r+1}\rho_1+m\left(\frac{r+1-\theta}{r+1}\right)\frac{\rho_2}{m}\right)+\left(1-h\left(\frac{1}{2}\right)\right)^s\chi^{(n)}\left(\frac{\theta}{r+1}\rho_2+m\left(\frac{r+1-\theta}{r+1}\right)\frac{\rho_1}{m}\right)\leqslant \\
&h^s\left(\frac{1}{2}\right)\left[\chi^{(n)}(\rho_1)\,h^s\left(\frac{\theta}{r+1}\right)+m\chi^{(n)}\left(\frac{\rho_2}{m}\right)\left(1-h\left(\frac{r+1-\theta}{r+1}\right)\right)^s\right]+ \\
&\left(1-h\left(\frac{1}{2}\right)\right)^s\left[\chi^{(n)}(\rho_2)\,h^s\left(\frac{\theta}{r+1}\right)+m\chi^{(n)}\left(\frac{\rho_1}{m}\right)\left(1-h\left(\frac{r+1-\theta}{r+1}\right)\right)^s\right].
\end{aligned}$$

Multiplying by $w(\theta)$, integrating with respect to the variable θ between 0 and 1, we obtain the right side of (2.1). In this way, the proof is complete. □

The following result would be very useful to prove future theorems.

Lemma 2.2. *Let χ be a real function defined on the real interval $[\rho_1, \rho_2]$ and differentiable on (ρ_1, ρ_2). If $\chi' \in L_1(\rho_1, \rho_2)$, and $w(\theta)$ is a function differentiable on (ρ_1, ρ_2), then we have*

$$\begin{aligned} &\left\{-w(1)\left(\chi^{(n)}\left(\frac{\rho_1+r\rho_2}{r+1}\right)+\chi^{(n)}\left(\frac{r\rho_1+\rho_2}{r+1}\right)\right)+w(0)\left(\chi^{(n)}(\rho_1)+\chi^{(n)}(\rho_2)\right)\right\} \\ +& \frac{r+1}{\rho_2-\rho_1}\left[\left({}^{C}_{n}D^{w'}_{\left(\frac{r\rho_1+\rho_1}{r+1}\right)-}\chi\right)(\rho_1)+\left({}^{C}_{n}D^{w'}_{\left(\frac{\rho_1+r\rho_1}{r+1}\right)+}\chi\right)(\rho_2)\right] \\ =& \frac{\rho_2-\rho_1}{r+1}\int_0^1 w(\theta)\left[\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)-\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2+\frac{r+1-\theta}{r+1}\rho_1\right)\right]d\theta. \end{aligned} \tag{2.3}$$

Proof. First of all, let's see that

$$\begin{aligned} &\int_0^1 w(\theta)\left[\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)-\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2+\frac{r+1-\theta}{r+1}\rho_1\right)\right]d\theta \\ =& \int_0^1 w(\theta)\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)d\theta-\int_0^1\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2+\frac{r+1-\theta}{r+1}\rho_1\right)d\theta \\ =& P_1-P_2. \end{aligned}$$

Using the integration by parts method, we have

$$P_1=\frac{r+1}{\rho_2-\rho_1}\left[-w(1)\chi^{(n)}\left(\frac{\rho_1+r\rho_2}{r+1}\right)+w(0)\chi^{(n)}(\rho_2)\right]+\frac{(r+1)^2}{(\rho_2-\rho_1)^2}\int_{\rho_1}^{\frac{r\rho_1+\rho_2}{r+1}} w'\left[\frac{x-\rho_1}{\frac{\rho_2-\rho_1}{r+1}}\right]\chi^{(n)}(x)dx,$$

because,

$$\int_0^1 w'(\theta)\chi^{(n)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)d\theta=\frac{n+1}{\rho_2-\rho_1}\int_{\rho_1}^{\frac{r\rho_1+\rho_2}{r+1}} w'\left[\frac{x-\rho_1}{\frac{\rho_2-\rho_1}{r+1}}\right]\chi^{(n)}(x)dx.$$

Similarly,

$$P_2=\frac{r+1}{\rho_2-\rho_1}\left[w(1)\chi^{(n)}\left(\frac{r\rho_1+\rho_2}{r+1}\right)-w(0)\chi^{(n)}(\rho_1)\right]-\frac{(r+1)^2}{(\rho_2-\rho_1)^2}\int_{\frac{\rho_1+r\rho_2}{r+1}}^{\rho_2} w'\left[\frac{\rho_2-x}{\frac{\rho_2-\rho_1}{r+1}}\right]\chi^{(n)}(x)dx.$$

From P_1-P_2, we obtain the desired equality. □

The result obtained in lemma 2.2 will be necessary to prove the following theorems.

Theorem 2.3. *Let χ be a positive real function defined on $[\rho_1,\rho_2]\subset\mathbb{R}$, such that $\chi^{(n)}\in L_1(\rho_1,m\rho_2)$. If $|\chi^{(n)}|$ es (h,m)-modified convex of the second type in $\left[\rho_1,\frac{\rho_2}{m}\right]$, then we have*

$$\mathbb{I}(\rho_1,\rho_2,\chi,w) \leqslant \frac{\rho_2-\rho_1}{r+1}\left(|\chi^{(n+1)}(\rho_1)|+|\chi^{(n+1)}(\rho_2)|\right)\mathbb{G}+m\left(|\chi^{(n+1)}\left(\frac{\rho_1}{m}\right)|+|\chi^{(n+1)}\left(\frac{\rho_2}{m}\right)|\right)\mathbb{H} \tag{2.4}$$

where $\mathbb{I}(\rho_1,\rho_2,\chi,w)$ is the absolute value of the left side of (2.3),

$$\mathbb{G}=\int_0^1|w(\theta)|h^s\left(\frac{\theta}{r+1}\right)d\theta,$$

and

$$\mathbb{H}=\int_0^1|w(\theta)|\left(1-h\left(\frac{r+1-\theta}{r+1}\right)\right)^s d\theta.$$

Proof. From Lemma 2.2, we have

$$
\begin{aligned}
&\left|\int_0^1 w(\theta)\left[\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)-\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2+\frac{r+1-\theta}{r+1}\rho_1\right)\right]d\theta\right| \\
\leqslant\ & \int_0^1 |w(\theta)|\left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)\right|d\theta+\int_0^1 |w(\theta)|\left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2+\frac{r+1-\theta}{r+1}\rho_1\right)\right|d\theta.
\end{aligned}
$$

By the modified (h, m)-convexity of $|\chi^{(n+1)}|$, we get

$$
\begin{aligned}
&\int_0^1 |w(\theta)|\left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)\right|d\theta \\
\leqslant\ & \int_0^1 |w(\theta)|\left[h^s\left(\frac{\theta}{r+1}\right)|\chi^{(n+1)}(\rho_1)|+m\left(1-h\left(\frac{r+1-\theta}{r+1}\right)\right)^s\left|\chi^{(n+1)}\left(\frac{\rho_2}{m}\right)\right|\right]d\theta \\
=\ & |\chi^{(n+1)}(\rho_1)|\int_0^1 |w(\theta)|h^s\left(\frac{\theta}{r+1}\right)d\theta+m\left|\chi^{(n+1)}\left(\frac{\rho_2}{m}\right)\right|\int_0^1 |w(\theta)|\left(1-h\left(\frac{r+1-\theta}{r+1}\right)\right)^s d\theta. \quad (2.5)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_0^1 |w(\theta)|\left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2+\frac{r+1-\theta}{r+1}\rho_1\right)\right|d\theta \\
\leqslant\ & |\chi^{(n+1)}(\rho_2)|\int_0^1 |w(\theta)|h^s\left(\frac{\theta}{r+1}\right)d\theta+m\left|\chi^{(n+1)}\left(\frac{\rho_1}{m}\right)\right|\int_0^1 |w(\theta)|\left(1-h\left(\frac{r+1-\theta}{r+1}\right)\right)^s d\theta.
\end{aligned}
$$

From (2.5) and (2.6), we obtain (2.4). □

We can improve the previous result if we impose additional conditions on $|\chi^{(n+1)}|^q$, as in the following theorem.

Theorem 2.4. *Let χ be a positive real function defined on $[\rho_1, \rho_2] \subset \mathbb{R}$, such that $\chi^{(n+1)} \in L_1(\rho_1, m\rho_2)$. If $|\chi^{(n+1)}|^q$ is a modified convex function of the second type on $\left[\rho_1, \frac{\rho_2}{m}\right]$, then we have*

$$
\mathbb{I}(\rho_1,\rho_2,\chi,w) \leqslant \frac{\rho_2-\rho_1}{r+1}\mathbb{J}_p\sum_{i=1}^{2}\left[|\chi^{(n+1)}(\rho_{3-i})|\mathbb{K}+m\left|\chi^{(n+1)}\left(\frac{\rho_i}{m}\right)\right|\mathbb{L}\right]^{\frac{1}{q}}, \quad (2.6)
$$

where $\mathbb{J}_p=\left(\int_0^1 |w(\theta)|^p d\theta\right)^{\frac{1}{p}}$, $\mathbb{K}=\int_0^1 h^s\left(\frac{\theta}{r+1}\right)d\theta$ *and* $\mathbb{L}=\int_0^1\left(1-h\left(\frac{r+1-\theta}{r+1}\right)\right)^s d\theta$.

Proof. Analogously to Theorem 2.3, we have

$$
\begin{aligned}
&\left|\int_0^1 w(\theta)\left[\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)-\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2+\frac{r+1-\theta}{r+1}\rho_1\right)\right]d\theta\right| \\
\leqslant\ & \int_0^1 |w(\theta)|\left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)\right|d\theta+\int_0^1 |w(\theta)|\left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2+\frac{r+1-\theta}{r+1}\rho_1\right)\right|d\theta.
\end{aligned}
$$

Now, from Hölder's inequality, we have

$$
\begin{aligned}
&\int_0^1 |w(\theta)|\left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)\right|d\theta \\
\leqslant\ & \left(\int_0^1 |w(\theta)|^p d\theta\right)^{\frac{1}{p}}\left(\int_0^1\left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)\right|^q d\theta\right)^{\frac{1}{q}}, \quad (2.7)
\end{aligned}
$$

and

$$\begin{aligned}&\int_0^1 |w(\theta)| \left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2+\frac{r+1-\theta}{r+1}v1\right)\right| d\theta \\ &\leqslant \left(\int_0^1 |w(\theta)|^p d\theta\right)^{\frac{1}{p}} \left(\int_0^1 \left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2+\frac{r+1-\theta}{r+1}\rho_1\right)\right|^q d\theta\right)^{\frac{1}{q}}, \end{aligned} \tag{2.8}$$

for $\frac{1}{p}+\frac{1}{q}=1$. Using the (h,m)-convexity of the second kind of $|\chi^{(n+1)}|^q$, we obtain from (2.7) and (2.8);

$$\begin{aligned}&\int_0^1 \left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)\right|^q d\theta \\ &\leqslant \left|\chi^{(n+1)}(\rho_1)\right|^q \int_0^1 h^s\left(\frac{\theta}{r+1}\right) d\theta + m\left|\chi^{(n+1)}\left(\frac{\rho_2}{m}\right)\right|^q \int_0^1 \left(1-h\left(\frac{r+1-\theta}{r+1}\right)\right)^s d\theta, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned}&\int_0^1 \left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2+\frac{r+1-\theta}{r+1}\rho_1\right)\right|^q d\theta \\ &\leqslant \left|\chi^{(n+1)}(\rho_2)\right|^q \int_0^1 h^s\left(\frac{\theta}{r+1}\right) d\theta + m\left|\chi^{(n+1)}\left(\frac{\rho_1}{m}\right)\right|^q \int_0^1 \left(1-h\left(\frac{r+1-\theta}{r+1}\right)\right)^s d\theta. \end{aligned} \tag{2.10}$$

Substituting (2.9), (2.10) into (2.7) and (2.8), we obtain the desired inequality. □

Theorem 2.5. *Let χ be a positive real function defined on $[\rho_1,\rho_2]\subset\mathbb{R}$, such that $\chi^{(n+1)}\in L_1(\rho_1, m\rho_2)$. If $|\chi^{(n+1)}|^q$, $q>1$, is a modified (h,m)-convex function of the second type in $[\rho_1,\frac{\rho_2}{m}]$, then we have*

$$\mathbb{I}(\rho_1,\rho_2,\chi,w) \leqslant \frac{\rho_2-\rho_1}{r+1}\mathbb{J}_q\sum_{i=1}^{2}\left[\left(|\chi^{(n+1)}(\rho_{3-i})|^q\mathbb{G}+m|\chi^{(n+1)}\left(\frac{\rho_i}{m}\right)|^q\mathbb{H}\right)^{\frac{1}{q}}\right], \tag{2.11}$$

where $\mathbb{J}_q=\left(\int_0^1|w(\theta)|d\theta\right)^{1-\frac{1}{q}}$.

Proof. We know that

$$\begin{aligned}&\left|\int_0^1 w(\theta)\left[\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)-\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2+\frac{r+1-\theta}{r+1}\rho_1\right)\right]d\theta\right| \\ &\leqslant \int_0^1 |w(\theta)||\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)|d\theta+\int_0^1 |w(\theta)||\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2+\frac{r+1-\theta}{r+1}\rho_1\right)|d\theta. \end{aligned}$$

From the mean power inequality, we have

$$\begin{aligned}&\int_0^1 |w(\theta)| \left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)\right| d\theta \\ &\leqslant \left(\int_0^1 |w(\theta)| d\theta\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1+\frac{r+1-\theta}{r+1}\rho_2\right)\right|^q d\theta\right)^{\frac{1}{q}}, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned}&\int_0^1 |w(\theta)| \left| \chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2 + \frac{r+1-\theta}{r+1}\rho_1\right)\right| d\theta \\ \leqslant &\left(\int_0^1 |w(\theta)| d\theta\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2 + \frac{r+1-\theta}{r+1}\rho_1\right)\right|^q d\theta\right)^{\frac{1}{q}}. \end{aligned} \tag{2.13}$$

Using the modified (h, m)-convexity of $|\chi^{(n+1)}|^q$, we obtain

$$\int_0^1 \left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_1 + \frac{r+1-\theta}{r+1}\rho_2\right)\right|^q d\theta \leqslant |\chi^{(n+1)}(\rho_1)|^q \mathbb{G} + m\|\chi^{(n+1)}\left(\frac{\rho_2}{m}\right)|^q \mathbb{H}, \tag{2.14}$$

and

$$\int_0^1 \left|\chi^{(n+1)}\left(\frac{\theta}{r+1}\rho_2 + \frac{r+1-\theta}{r+1}\rho_1\right)\right|^q d\theta \leqslant |\chi^{(n+1)}(\rho_2)|^q \mathbb{G} + m|\chi^{(n+1)}\left(\frac{\rho_1}{m}\right)|^q \mathbb{H}. \tag{2.15}$$

Substituting (2.14) and (2.15) into (2.12) and (2.13) respectively, we arrive at (2.11). □

Remark 2.6. All the previous results contain many of those reported in the literature, particularizing the function w as well as for different notions of convexity, for example, those obtained in [1, 2, 3, 4, 5] and [6].

References

[1] B. Bayraktar, J. E. Nápoles V., *A note on Hermite-Hadamard integral inequality for (h,m) - convex modified functions in a generalized framework*, submited. 1, 2.6

[2] G. Farid, A. Javed, A. U. Rehman, M. I. Qureshi, *On Hadamard-type inequalities for differentiable functions via Caputo k-fractional derivatives*, Cogent Mathematics (2017), 4: 1355429, https://doi.org/10.1080/23311835.2017.1355429. 2.6

[3] G. Farid, S. Naqvi, A. U. Rehman, *A version of the Hadamard inequality for Caputo fractional derivatives and related results*, RGMIA Research Report Collection, 2017, 11 pp, 20, Article 59. 2.6

[4] P. M. Guzmán, J. E. Nápoles V., V. Stojiljković, *New extensions of the Hermite-Hadamard inequality*, Contrib. Math. 7 (2023) 60–66 DOI: 10.47443/cm.2023.032 2.6

[5] S. M. Kang, G. Farid, W. Nazeer, S. Naqvi, *A version of the Hadamard inequality for Caputo fractional derivatives and related results*, Journal of Computational Analysis & Applications 27 (6), 2019, 962–972. 2.6

[6] L. N. Mishra, Q. U. Ain, G. Farid,A. U. Rehman, *k -fractional integral inequalities for (h,m)- convex functions via Caputo k -fractional derivatives*, Korean J. Math. 27 (2019), no. 2, pp. 357-374, https://doi.org/10.11568/kjm.2019.27.2.357. 2.6